This paper provides a theory of credit-driven stock market bubbles. Our theory is motivated by two observations. First, the United States has experienced stock market booms and busts, which may not be explained entirely by fundamentals. Figure 1 presents the monthly real (adjusted for inflation using the Consumer Price Index) Standard and Poor's (S&P) Composite Stock Price Index from January 1871 to January 2011 (upper line), and the corresponding series of real S&P Composite earnings (lower line) for the same period. From this figure, we see that the most dramatic bull market in U.S. history is from July 1982 to August 2000, with the skyrocketing increase in the price index during the late 1990s being the most remarkable. The latter price increase is often attributed to the internet bubble. Yet, the dramatic rise in prices since 1982 is not matched in real earnings growth. The lower line in Figure 1 shows that earnings seem to be oscillating around a slow, steady growth path that has persisted for over a century. Following the peak in 2000, the stock market crashed, reaching the bottom in February 2003. After then the stock market went up and reaching the peak in October 2007. This stock market runup is often attributed to the housing market bubble. Following the burst of the bubble, the U.S. economy has entered the Great Recession, with the stock market drop of 51.7% from October 2007 until March 2009. The recent stock market behavior resembles the runup of the 1920s (the Roaring Twenties), culminating in the 1929 crash. The stock market moves during that period are often used as an example of bubbles and crashes.

Second, some episodes of stock market booms are accompanied by credit booms. This suggests that one possible cause of bubbles is excessive liquidity in the financial system, inducing lax or inappropriate lending standards by the banks.[1] Figure 2 presents the S&P price index in relation to two credit market condition indexes for the United States. Both panels in the figure shows that since 2003 until then, the stock market boom was associated with credit ease and the bust was associated with credit tightening. However, this relationship did not exist during 1990s. This figure suggests that the recent stock market bubble and crash may be credit driven, while it is not the case for the internet bubble and crash.

The above two observations can be applied to many other countries, especially in emerging market countries. For example, overoptimism in 1990s about an "East Asian miracle" gen-

[1] For example, Axel A. Weber, the former president of the Deutsche Bundesbank, has argued that "The past has shown that an overly generous provision of liquidity in global financial markets in connection with a very low level of interest rates promotes the formation of asset-price bubbles." (http://www.bloomberg.com/apps/news?pid=newsarchive&sid=a5S5Boes29Io)

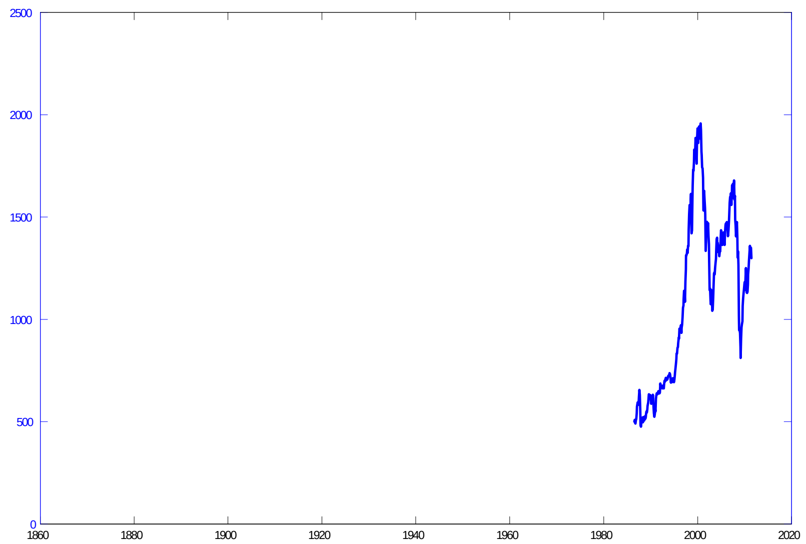

Figure 1: This figures plots the S&P Price Index and earnings. The solid line represents the S&P Price Index and the dashed line represents earnings. The data are downloaded from Robert Shiller's website: http://www.econ.yale.edu/~shiller/data.htm

erated high economic growth in East Asian countries. Capital account and financial market liberalization contributed to large capital inflows and generated a lending boom. The rapid increase in asset prices including housing prices and stock prices were accompanied by a large expansion of domestic credit through under-regulated banking systems. As documented by Collyns and Senhadji (2002), regional stock markets (including Hong Kong, Indonesia, Korea, Malaysia, Philippines, Singapore, Taiwan, and Thailand) trended upward through the first part of the 1990s, generally peaking around 1997 at an average 165% higher than their value in January 1991. Regional stock prices fell sharply after the onset of the crisis in mid-1997 through the end of 1998. In particular, Korea, Malaysia, and Thailand all suffered declines of over 70%.

To formalize our theory, we construct a tractable model of a production economy in which households are infinitely lived and trade firm stocks. We assume that households have linear utility so that the interest rate is equal to the constant subjective discount rate. There is no aggregate uncertainty.[2] A continuum of firms meet idiosyncratic stochastic investment

[2] These two assumptions are adopted for simplicity. Miao and Wang (2011a) introduce a concave utility

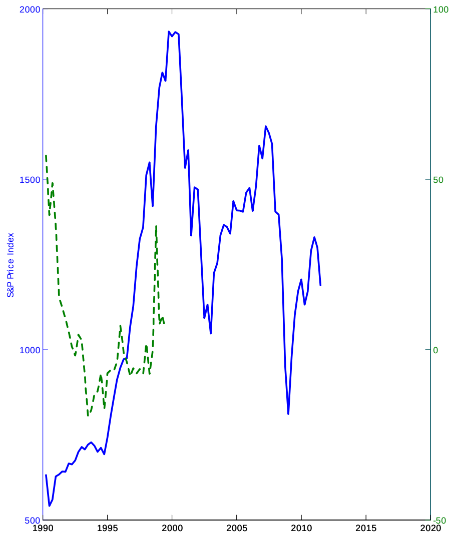

Figure 2: This …gure plots S&P Price Index and two credit market condition indexes. The solid lines in the two panels represent the S&P Price Index. The dashed line on the left panel represents Net Percentage of Domestic Respondents Tightening Standards for Commercial and Industrial Loans Large and Medium Firms. The dashed line on the right panel represents the St. Louis Financial Stress Index. The last two series are downloaded from the Federal Reserve Bank of St. Louis.

opportunities as in Kiyotaki and Moore (1997, 2005, 2008) and face credit constraints. We model credit constraints in a way similar to that in Kiyotaki and Moore (1997), Albuquerque and Hopenhayn (2004), and Jermann and Quadrini (2010).[3] Specifically, durable assets (or capital in our model) are used not only as inputs for production, but also as collateral for loans. Borrowing is limited by the market value of the collateral. Unlike Kiyotaki and Moore (1997) who assume that the market value of the collateral is equal to the liquidation value of the collateralized assets, we assume that it is equal to the of the reorganized firm with these assets. Because the going-concern value is priced in the stock market, it may contain a bubble component. If both lenders and the credit-constrained borrowers (firms in our model) optimistically believe that the collateral value is high possibly because of bubbles, firms will want to borrow more and lenders won't mind lending more. Consequently, firms can finance more investment and accumulate more assets for future production, making their assets indeed more valuable.[4] This positive feedback loop mechanism makes the lenders' and the borrowers' beliefs self-fulfilling and bubbles may sustain in equilibrium. We refer to this equilibrium as the bubbly equilibrium.

Of course, there is another equilibrium in which no one believes in bubbles and hence bubbles do not appear. We call this equilibrium the bubbleless equilibrium. We provide explicit conditions to determine which type of equilibrium can exist. We show that if the degree of pledgeability is sufficiently small, then both bubbleless and bubbly equilibria can exist; otherwise, only the bubbleless equilibrium exists. This result is intuitive. If the degree of pledgeability is sufficiently small, investors have incentives to inflate their asset values to relax the collateral constraint and bubbles may emerge. The emergence of bubbles is accompanied by a credit boom. If the degree of pledgeability is sufficiently large, investors can borrow enough to finance investment. There is no need for them to create bubbles.

We prove that the bubbly equilibrium has two steady states: a bubbly one and a bubbleless one. Both steady states are inefficient due to credit constraints and both are local saddle points. Thus, multiple equilibria in our model are not generated by indeterminacy as in the literature

production. Wang and Wen (2011) provide a model similar to that in Kocherlakota (2009). They study asset price volatility and bubbles that may grow on assets with exogenous rents. They assume that these assets cannot be used as an input for production. Our model can also generate bubbles on intrinsically useless assets as long as these assets can be used to finance investment and households face short sales constraints. These assumptions are standard in the literature (e.g., Kocherlakota (2009) and Wang and Wen (2011)).

Building on Diamond (1965) and Tirole (1985), Caballero and Krishnamurthy (2006), Farhi and Tirole (2010), and Martin and Ventura (2010a,b) study bubbles in overlapping generations models with credit constraints. Caballero and Krishnamurthy (2006) show that stochastic bubbles are beneficial because they provide domestic stores of value, thereby reducing capital outflows while increasing investment. But they come at a cost, as they expose the country to

Appendix D analyzes a discrete-time setup when firms face idiosyncratic investment-specific shocks with a continuous distribution.

where w_t is the wage rate and

$$R_t = \frac{w_t}{1}^{\frac{-1}{}} : \tag{2}$$

We will show later that R_t is equal to the marginal product of capital or the rental rate of capital.

Following Kiyotaki and Moore (1997, 2005, 2008), we assume that each …rm j meets an opportunity to make investment in capital with probability dt in period t. With probability 1 dt; no investment opportunity arrives. Thus, capital evolves according to:

$$K^j_{t+dt} = \begin{cases} (1 \quad dt)\, K^j_t + I^j_t & \text{with probability } \quad dt \\ (1 \quad dt)\, K^j_t & \text{with probability } 1 \quad dt \end{cases} ; \tag{3}$$

where > 0 is the depreciation rate of capital and I^j_t is the investment level. This assumption captures …rm-level investment lumpiness and generates ex post …rm heterogeneity. Assume that the arrival of the investment opportunity is independent across …rms and over time. In Appendix D, we study the case where …rms are subject to idiosyncratic investment-speci…c shocks with a continuous distribution. This alternative modeling does not change our key insights.

Let the ex ante …rm value (or stock value) prior to observing the arrival of investment opportunities be $V_t(K^j_t)$; where we suppress aggregate state variables in the argument. It satis…es the following Bellman equation:

$$\begin{aligned} V_t(K^j_t) \quad = \quad & \max_{I^j_t} \; R_t K^j_t dt \quad I^j_t dt + e^{\;rdt} V_{t+dt}((1 \quad dt)\, K^j_t + I^j_t) \;\; dt \\ & + e^{\;rdt} V_{t+dt}((1 \quad dt)\, K^j_t)(1 \quad dt); \end{aligned} \tag{4}$$

subject to some constraints on investment to be speci…ed next. As will be shown in Section 3, the optimization problem in (4) is not well de…ned if there is no constraint on investment given our winuti386(i)1aves starmn 9(e)ingdu333(ev)28ondhe-18.43sbutimng rD, wn our94(i)1ou4

We now write the investment constraint as:

$$0 \le I_t^j \le R_t K_t^j + L_t^j, \tag{5}$$

where $R_t K_t^j$ represents internal funds and L_t^j represents loans from financial intermediaries. To reduce the number of state variables and keep the model tractable, we consider intratemporal loans as in Carlstrom and Fuerst (1997) and Jermann and Quadrini (2010). These loans are taken at the beginning of the period and repaid at the end of the period. They do not have interests. In Appendix C, we incorporate intertemporal bonds with interest payments and allow firms to save. We show that our key insights and analysis carry over to this setup.

The key assumption of our model is that loans are subject to the collateral constraint:

$$L_t^j \le e^{-rdt} V_{t+dt}(\xi K_t^j). \tag{6}$$

The motivation of this constraint is similar to that in Kiyotaki and Moore (1997): Firm j pledges a fraction $\xi \in (0,1]$ of its assets (capital stock) K_t^j at the beginning of period t as the collateral. The parameter ξ may represent the tightness of the collateral constraint or the extent of financial market imperfections. It is the key parameter for our analysis below. At the end of period t, the stock market value of the collateral is equal to $e^{-rdt} V_{t+dt}(\xi K_t^j)$. The lender never allows the loan repayment L_t^j to exceed this value. If this condition is violated, then firm j may take loans L_t^j and walk away, leaving the collateralized asset ξK_t^j behind. In this case, the lender runs the firm with the collateralized assets ξK_t^j at the beginning of period $t + dt$ and obtains the smaller firm value $e^{-rdt} V_{t+dt}(\xi K_t^j)$ at the end of period t, which is the collateral value. Alternatively, the lender sells the collateralized assets to a third party at the value $e^{-rdt} V_{t+dt}(\xi K_t^j)$. The third party runs the firm with these assets and obtains the $e^{-rdt} V_{t+dt}(\xi K_t^j)$. Implicitly, we assume that firm assets are not specific to a particular owner. Any owner can operate the assets using the same technology.

We may interpret the collateral constraint in (6) as an incentive constraint in an optimal contract between firm j and the lender with limited commitment.[9] Given a history of information at date t, the beginning of period t; and repayments L_t^j at the end of period t; only when an investment opportunity arrives with Poisson probability dt: When no investment opportunity arrives, the firm does not invest and hence does not borrow. Firm j may default on debt at the end of period t. If it happens, then the firm and the lender renegotiate the loan repayment. In addition, the lender reorganizes the firm. Because of default costs, the lender can only seize a fraction of capital K_t^j: Alternatively, we may interpret as an efficiency parameter in that the lender may not be able to efficiently use the firm's assets K_t^j: The lender can run the firm with these assets at the beginning of period t + dt and obtains firm value $e^{\ rdt}V_{t+dt}(\ K_t^j)$. Or it can sell these assets to a third party at the going-concern value $e^{\ rdt}V_{t+dt}(\ K_t^j)$ if the third party can run the firm using assets K_t^j at the beginning of period t + dt. This value is the threat value (or the collateral value) to the lender at the end of period t. Following Jermann and Quadrini (2010), we assume that the firm has all the bargaining power in the renegotiation and the lender gets only the threat value. The key difference between our modeling and that of Jermann and Quadrini (2010) is that the threat value to the lender is the going concern value in our model, while Jermann and Quadrini (2010) assume that the lender liquidates the firm's assets and obtains the liquidation value in the even of default.[10]

Enforcement requires that, when the investment opportunity arrives at date t; the continuation value to the firm of not defaulting is not smaller than the continuation value of defaulting, that is,

$$\begin{aligned} & e^{\ rdt}V_{t+dt}((1\ \ dt)K_t^j + I_t^j)\ \ L_t^j \\ & e^{\ rdt}V_{t+dt}((1\ \ dt)K_t^j + I_t^j)\ \ e^{\ rdt}V_{t+dt}(\ K_t^j): \end{aligned} \tag{7}$$

This incentive constraint is equivalent to the collateral constraint in (6). This constraint ensures that there is no default in an optimal contract.

In the continuous-time limit, the collateral constraint becomes

$$L_t^j\ \ V_t(\ K_t^j): \tag{8}$$

Note that our modeling of collateral constraint is different from that of Kiyotaki and Moore (1997). We may write the Kiyotaki-Moore-type collateral constraint in our continuous-time framework as:

$$L_t^j\ \ Q_t K_t^j; \tag{9}$$

[10] U.S. Bankruptcy law has recognized the need to preserve going concern value when reorganizing businesses in order to maximize recoveries by creditors and shareholders (see 11 U.S.C. 1101 et seq.). Bankruptcy laws seek to preserve going concern value whenever possible by promoting the reorganization, as opposed to the liquidation, of businesses.

where Q_t represents the shadow price of capital. The expression $Q_t K_t^j$ is the shadow value of the collateralized assets or the liquidation value.[11] In Section 5, we shall argue that this type of collateral constraint will rule out bubbles. By contrast, according to (6), we allow the collateralized assets are valued in the stock market as the going-concern value when the firm is reorganized and keeps running using the collateralized assets after default. If both the firm and the lender believe that the firm's assets may be overvalued due to stock market bubbles, then these bubbles will relax the collateral constraint, which provides a positive feedback loop mechanism.

Let $K_t = \int_0^1 K_t^j dj$; $I_t = \int_0^1 I_t^j dj$; $N_t = \int_0^1 N_t^j dj$; and $Y_t = \int_0^1 Y_t^j dj$ be the aggregate capital stock, the aggregate investment, the aggregate labor demand, and aggregate output. Then a competitive equilibrium is defined as sequences of $\{Y_t\}$; $\{C_t\}$; $\{K_t\}$, $\{I_t\}$; $\{N_t\}$; $\{w_t\}$; $\{R_t\}$; $\{V_t(K_t^j)\}$; $\{I_t^j\}$; $\{K_t^j\}$; $\{N_t^j\}$ and $\{L_t^j\}$ such that households and firms optimize and markets clear in that:

$$\begin{aligned} N_t &= 1; \\ C_t + I_t &= Y_t; \\ K_{t+dt} &= (1 - \delta dt) K_t + I_t dt. \end{aligned}$$

We first solve an individual firm's optimal contract problem (4) subject to (3), (5), and (6) when the wage rate w_t or the rental rate R_t in (2) is taken as given. This problem does not give a contraction mapping and hence may admit multiple solutions. We conjecture that ex ante firm value takes the following form:

$$V_t(K_t^j) = v_t K_t^j + b_t; \tag{10}$$

where v_t and b_t are to be determined and depend on aggregate states only. Note that $b_t = 0$ is a possible solution. In this case, we may interpret $v_t K_t^j$ as the fundamental value of the firm.

[11] Note that our model differs from the Kiyotaki and Moore model in market arrangements, besides other specific modeling details. Kiyotaki and Moore assume that there is a market for physical capital (corresponding to land in their model), but there is no stock market for trading firm shares. In addition, they assume that households and entrepreneurs own firms and trade physical capital in the capital market. By contrast, we assume that households trade firm shares in the stock market and that firms own physical captial and make investment

The fundamental value is proportional to the firm's assets K_t^j; which has the same form as that derived in Hayashi (1982). The firm has no fundamental value if it has no assets ($K_t^j = 0$): There may be another solution in which $b_t > 0$: In this case, we interpret b_t as a bubble.[12]

Let Q_t be the Lagrange multiplier associated with the constraint (3) if the investment opportunity arrives. It represents the shadow price of capital or Tobin's marginal Q. The following

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t= 041099 222720[(0)]J5798 5128[()]J41099 163m[(7)]J5791 004553070[()]J.897[()]J51099 7476 70[(=)]38 041099 2227201

…rm will make the minimal possible level of investment. This investment choice is similar to Tobin's Q

$$Q_t = e^{\quad rdt}[R_{t+dt}dt + (1 \quad dt)Q_{t+dt} + (R_{t+dt} + \quad Q_{t+dt})(Q_{t+dt} \quad 1) \quad dt]; \tag{17}$$

$$K_{t+dt} = (1 \quad dt)K_t + \quad (R_t K_t + \quad Q_t K_t + B_t)\, dt; \;\; K_0 \tag{18}$$

$$\lim_{T\,!\,1} e^{\quad rTdt} Q_T K_{T+dt} = 0 \qquad \lim_{T\,!\,1} e^{\quad rTdt} B_T = 0;$$

$R_t = \quad K_t^{\quad 1}:$

When dt = 1, the above system reduces to the usual discrete-time characterization of equilibrium. However, this system is not convenient for analytically characterizing local dynamics. We may solve this system numerically by assigning parameter values. Instead of pursuing this route, we use analytical methods in the continuous-time limit as dt goes to zero. To compute the limit, we use the heuristic rule $dX_t = X_{t+dt} \quad X_t$ for any variable X_t: We also use the notation $\dot{X}_t = dX_t = dt$: We obtain the following:

$Q_t > 1$: dt ! 0;

$(B_t; Q_t; K_t)$

$$\dot{B}_t = rB_t \quad B_t \; (Q_t \quad 1); \tag{19}$$

$$\dot{Q}_t = (r + \quad) Q_t \quad R_t \qquad (R_t + \quad Q_t)(Q_t \quad 1); \tag{20}$$

$$\dot{K}_t = \qquad K_t + \quad (R_t K_t + \quad Q_t K_t + B_t); \;\; K_0 \tag{21}$$

$$\lim_{T\,!\,1} e^{\quad rT} Q_T K_T = 0 \qquad \lim_{T\,!\,1} e^{\quad rT} B_T = 0;$$

$R_t = \quad K_t^{\quad 1}$ $\qquad Q_t = v_t \qquad B_t = b_t \qquad j$

$V_t(K_t^j) = Q_t K_t^j + B_t$:

After obtaining the solution for $(B_t; Q_t; K_t)$; we can derive the equilibrium wage rate $w_t = (1 \quad) K_t$, the rental rate $R_t = \quad K_t^{\quad 1}$; aggregate output $Y_t = K_t$; aggregate investment,

$$I_t = R_t K_t + \quad Q_t K_t + B_t; \tag{22}$$

and aggregate consumption $C_t = Y_t \quad I_t$: We focus on two types of equilibrium.[13] The …rst type is bubbleless, for which $B_t = 0$ for all t: In this case, the market value of …rm j is equal to its fundamental value in that $V_t(K_t^j) = Q_t K_t^j$. The second type is bubbly, for which $B_t > 0$ for some t: We assume that assets can be freely disposed of so that the bubbles B_t cannot be negative. In this case, …rm value contains a bubble component in that $V_t(K_t^j) = Q_t K_t^j + B_t$ with $B_t > 0$: We next study these two types of equilibrium.

In a bubbleless equilibrium, $B_t = 0$ for all t: Equation (19) becomes an identity. We only need to focus on $(Q_t; K_t)$ determined by the di¤erential equations (20) and (21) in which $B_t = 0$ for all t. In the continuous time limit, $v_t = Q_t$

We …rst analyze the steady state. In the steady state, all aggregate variables are constant over time so that $Q_t = K_t = 0$. We use X to denote the steady state value of any variable X_t: By (20) and (21), we obtain the following steady-state equations:

$$0 = (r + \ )Q \quad R \qquad (R + \ Q)(Q \quad 1); \tag{23}$$

$$0 = \qquad K + \ (RK + \ QK): \tag{24}$$

We use a variable with an asterisk to denote its value in the bubbleless equilibrium. Solving equations (23)-(24) yields:

$$\frac{(1 \qquad)}{} \quad r; \tag{25}$$

$$Q_t = Q_E \quad 1 \qquad K_t = K_E;$$

K_E $\qquad\qquad$ $(K_E) \quad ^1 = r + \ :$

$$0 < \quad < \frac{(1 \qquad)}{} \quad r; \tag{26}$$

$$Q \quad = \frac{(1 \qquad)}{} \frac{1}{r +} > 1; \tag{27}$$

$$(K \ ) \quad ^1 \quad = \frac{(1 \qquad)}{} \frac{r}{r +} + \ : \tag{28}$$

$K \ < K_E$:

Assumption (25) says that if …rms pledge su¢ cient assets as the collateral, then the collateral constraints will not bind in equilibrium. The competitive equilibrium allocation is the same

the bubbleless equilibrium. This means that the bubbleless equilibrium must be dynamically ine¢ cient (see Tirole (1985)).

Unlike this literature, bubbles in our model are on reproducible real assets and also in‡uence their fundamentals (or dividends). Speci…cally, each unit of the bubble raises the collateral value by one unit and hence allows the …rm to borrow an additional unit. The …rm then makes one more unit of investment when an investment opportunity arrives. This unit of investment raises …rm value by Q_t: Subtracting one unit of costs, we then deduce that the second term on the left-hand side of (29) represents the net increase in …rm value for each unit of bubbles. This is why we call this term dividend yields. Dividend payouts make the growth rate of bubbles less than the interest rate. Thus, the transversality condition cannot rule out bubbles in our model. We can also show that the bubbleless equilibrium is dynamically e¢ cient in our model. Speci…cally, the golden rule capital stock is given by $K_{GR} = (=)^{\frac{1}{-1}}$: One can verify that $K < K_{GR}$: Thus, one cannot use the condition for the overlapping generations economies in Tirole (1985) to ensure the existence of bubbles. Below we will give new conditions to ensure the existence of bubbles in our model.

We …rst study the existence of a bubbly steady state in which $B > 0$: We use a variable with a subscript b to denote this variable's bubbly steady state value. By Proposition 3, $(B; Q_b; K_b)$ satis…es equations (23) and

$$0 = rB \quad B \ (Q \quad 1); \tag{30}$$

$$0 = \quad K + [RK + \ QK + B] \ : \tag{31}$$

Using these equations, we can derive:

$$\frac{B}{K_b} = - \quad \frac{r + \quad + \quad r +}{1 + r} \quad > 0; \tag{32}$$

$$Q_b = \frac{r}{} + 1 > 1; \tag{33}$$

$$(K_b) \quad ^{1} = \frac{(1 \quad)r +}{1 + r} \quad \frac{r}{} + 1 \quad ; \tag{34}$$

$$0 < \quad < \frac{(1 \quad)}{r +} \quad r: \tag{35}$$

$$Q_b < Q \; ; \qquad K_{GR} > K_E > K_b > K$$

only pledge a smaller amount of assets, they will face a tighter collateral constraint so that they have higher incentives to generate larger bubbles to finance investment.

Now, we study the stability of the two steady states and the local dynamics around these steady states. Since the equilibrium system (19)-(21) is three dimensional, we cannot use the phase diagram to analyze its stability. We thus consider a linearized system and obtain the following:

$(B; Q_b; K_b)$

$(0; Q \ ; K \ )$

More formally, in Appendix A, we prove that for the nonlinear system (19)-(21), there is a neighborhood $N \quad \mathbb{R}^3_+$ of the bubbly steady state $(B; Q_b; K_b)$ and a continuously differentiable function $\ : N \ ! \ \mathbb{R}^2$ such that given any K_0 there exists a unique solution $(B_0; Q_0)$ to the equation $\ (B_0; Q_0; K_0) = 0$ with $(B_0; Q_0; K_0)$ 2 N; and $(B_t; Q_t; K_t)$ converges to $(B; Q_b; K_b)$ starting at $(B_0; Q_0; K_0)$ as t approaches infinity. The set of points $(B; Q; K)$ satisfying the equation $\ (B; Q; K) = 0$ is a one dimensional stable manifold of the system. If the initial value $(B_0; Q_0; K_0)$ is on the stable manifold, then the solution to the nonlinear system (19)-(21) is also on the stable manifold and converges to $(B; Q_b; K_b)$ as t approaches infinity.

Although the bubbleless steady state $(0; Q \ ; K \ )$ is also a local saddle point, the local dynamics around this steady state are different. In Appendix A, we prove that the stable manifold for the bubbleless steady state is two dimensional. Formally, there is a neighborhood $N \quad \mathbb{R}^3_+$ of $(0; Q \ ; K \ )$ and a continuously differentiable function $\ : N \ ! \ \mathbb{R}$ such that given any $(B_0; K_0)$ there exists a unique solution Q_0 to the equation $\ (B_0; Q_0; K_0) = 0$ with $(B_0; Q_0; K_0)$ 2 N; and $(B_t; Q_t; K_t)$ converges to $(0; Q \ ; K \ )$ starting at $(B_0; Q_0; K_0)$ as t approaches infinity. Intuitively, along the two-dimensional stable manifold, the bubbly equilibrium is asymptotically bubbleless in that bubbles will burst eventually.

So far, we have focused on deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we now study stochastic bubbles. Consider the discrete-time economy described in Section 2. Suppose a bubble exists initially, $B_0 > 0$. In each time interval between t and

subject to

$$0 \quad I_t^j \quad R_t K_t^j + Q_t \ K_t^j + B_t; \tag{41}$$

where we de…ne $Q_t = e^{\ rdt} v_{t+dt}$;

$$Q_t = e^{\ rdt} \ (1 \quad dt) v_{t+dt} + \ v_{t+dt} dt \ ; \tag{42}$$

$$B_t = e^{\ rdt}(1 \quad dt) b_{t+dt}: \tag{43}$$

Suppose $Q_t > 1$: Then the optimal investment level achieves the upper bound in (41). Substituting this investment level into equation (40) and matching coe¢ cients on the two sides of this equation, we obtain:

$$v_t = R_t dt + Q_t(1 \quad dt) + \ (Q_t \quad 1)(R_t + Q_t \ ) dt; \tag{44}$$

$$b_t = B_t + \ (Q_t \quad 1) B_t dt: \tag{45}$$

As in Section 3, we conduct aggregation to obtain the discrete-time equilibrium system. We then take the continuous-time limits as $dt \ ! \ 0$ to obtain the following:

$Q_t > 1$:

$(B_t; Q_t; K_t)$

$$B_t = (r + \ ) B_t \quad (Q_t \quad 1) B_t; \tag{46}$$

$$Q_t = (r + \ + \ ) Q_t \quad Q_t \quad R_t \quad (Q_t \quad 1)(R_t + \ Q_t); \tag{47}$$

$$R_t = \ K_t^{\ 1} \qquad Q_t = g(K_t)$$

Equation (46) reveals that the rate of return on bubbles is equal to r + [illegible]

$g(K)$ after the bubble collapses and then converges to the bubbleless steady-state value Q given in equation (27).

Our objective is to show the existence of $(B_S; Q_S; K_S)$: By (46), we can show that

$$Q_S = \frac{r +}{} + 1: \qquad (48)$$

Since $Q_S > 1$, we can apply Proposition 7 in some neighborhood of Q_S: Equation (47) implies that

$$0 = (r + \quad + \quad)Q_S \quad g(K) \quad R \quad (Q_S \quad 1)(R + \quad Q_S); \qquad (49)$$

where $R = \quad K \quad ^{1}$: The solution to this equation gives K_S: Once we obtain K_S and Q_S; we use equation (31) to determine B_S:

The di¢ cult part is to solve for K_S: In doing so, we de…ne such that

$$\frac{r +}{} \; 1 \; 1 = \qquad 1$$

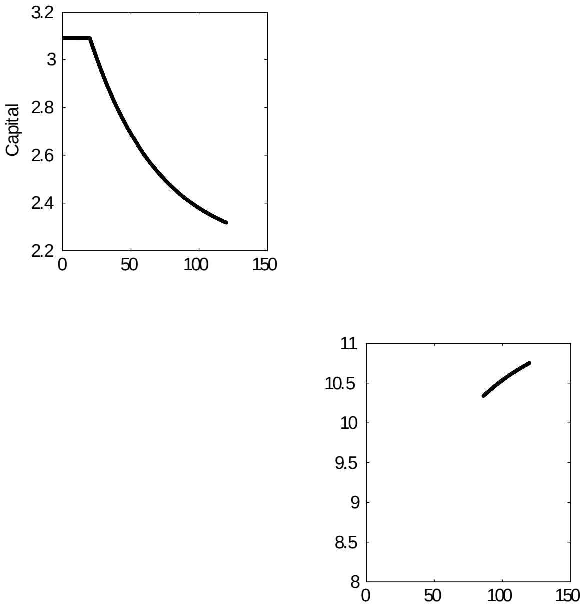

Figure 4: This ...gure plots the dynamics of the stationary equilibrium with stochastic bubbles. Assume that the bubble bursts at time t = 20: The parameter values are set as follows: r = 0:02; = 0:4; = 0:025; = 0:05; = 0:01; and :9214dβ

We have shown that the collapse of bubbles generates a recession. Is there a government policy that restores economic efficiency? The inefficiency of our model comes from the credit constraints. In our model, firms can use internal funds and external loans to finance investment. External loans are subject to collateral constraints. Bubbles help relax these constraints, while the collapse of bubbles tightens them.

Now suppose that the government can supply liquidity to the firms by issuing public bonds. These bonds are backed by lump-sum taxes.[18] We consider unbacked assets in Section 7.3. We first suppose that households can buy and sell these bonds without any trading frictions. We then relax this assumption in Section 7.3. Firms can use public bonds as collateral to relax their collateral constraints. They can also buy and sell these assets to finance investment. Let the quantity of government bonds supplied to the firms be M_t and the bond price be P_t. We start with the discrete-time environment described in Section 2. The value of the government assets satisfies:

$$M_t P_t = T_t dt + M_{t+dt} P_t, \tag{51}$$

where T_t denotes lump-sum taxes. Taking the continuous-time limits yields:

$$\dot{M}_t P_t = -T_t. \tag{52}$$

It is more convenient to define $D_t = P_t M_t$. Then use the fact that $\dot{D}_t = \dot{P}_t M_t + \dot{M}_t P_t$ to rewrite (52) as:

$$\dot{D}_t - \dot{P}_t M_t = \dot{D}_t - \frac{\dot{P}_t}{P_t} D_t = -T_t \text{ if } P_t > 0. \tag{53}$$

Since households are assumed to be risk neutral, we immediately obtain the asset pricing holdings of government bonds are another state variable. We write firm j's dynamic programming problem as follows:

$$V_t\left(K_t^j; M_t^j\right) = \max_{I_t^j;\, M_{t+dt}^j} R_t K_t^j dt - I_t^j dt + P_t\left(M_t^j - M_{t+dt}^j\right) \tag{55}$$
$$+ e^{-rdt} V_{t+dt}\left((1-\delta dt) K_t^j + I_t^j; M_{t+dt}^j\right) \pi dt$$
$$+ e^{-rdt} V_{t+dt}\left((1-\delta dt) K_t^j; M_{t+dt}^j\right)(1-\pi dt);$$

subject to (5) and

$$L_t^j \le e^{-rdt} V_{t+dt}\left(\xi K_t^j; 0\right) + P_t M_t^j; \tag{56}$$

$$M_{t+dt}^j \ge 0; \tag{57}$$

where M_t^j denotes the amount of government assets held by firm j: In equilibrium, $\int M_t^j dj = M_t$: Equation (56) indicates that firms use government assets as collateral. The expression $e^{-rdt} V_{t+dt}\left(\xi K_t^j; 0\right)$ gives the market value of the collateralized assets ξK_t^j: Equation (57) is a short sales constraint, which rules out Ponzi schemes for public assets. It turns out that this constraint will never bind for firms:431Td[(c)-3238.431Td[(sh)-As3(...in4(f)-1S(ct)-33ll)-on4(f)-13,3(wi)1(er)-34(f)

As in Proposition 2, we may conduct aggregation and derive the equilibrium system for $(P_t; Q_t; K_t)$ in the discrete time case. As in Proposition 3, the continuous-time limits satisfy the following differential equations:

$$\dot{P}_t = rP_t \quad P_t \ (Q_t \quad 1); \tag{61}$$

$$\dot{K}_t = \quad K_t + \ (R_t K_t + \ Q_t K_t + P_t M_t); \ K_0 \text{ given}, \tag{62}$$

and an equation analogous to (20) for Q_t. In addition, the transversality condition

$$\lim_{T \to \infty} e^{\ rT} P_T M_T = 0;$$

and other transversality conditions for K_t and B_t as described in Proposition 3 must be satisfied. Here, we omit the detailed derivation of these conditions and the above differential equations.

Equation (61) is an asset pricing equation for the firms' trading. It is identical to the asset pricing equation (19) for the bubble. This is because the government bonds and the bubble on firm assets play the same role for the firms in that both of them can be used to relax the credit constraints. The dividend yield of the government bonds to the firms is equal to $\ (Q_t \quad 1)$

$$\dot{K}_t = K_t + (R_t K_t + Q_t K_t + B_t + P_t M_t); \; K_0 \text{ given.} \tag{65}$$

By a no-arbitrage argument similar to that in the previous subsection, we deduce that $Q_t = 1$: By equation (47) and $Q_t = 1$; we deduce that $R_t = r +$; which gives the e¢ cient

(51) implies that M_t is constant over time since $T_t = 0$: We thus normalize $M_t = 1$ for all t: Our previous asset pricing equations (54) (61), and (64) for public bonds still apply here if households do not face trading frictions. Thus, if $P_t > 0$; then $Q_t = Q_t = 1$; which implies that $K_t = K_E$ for all t: In this case, the capital accumulation equations (62) and (65) imply that the public bond price P_t must be constant over time, contradicting with the asset pricing equations for bonds. Thus, in equilibrium $P_t = 0$: The intuition is that the public bond is a bubble when it is an unbacked asset. Its rate of return or its growth rate is equal to the riskfree interest rate which is higher than the zero economic growth rate. Thus, the bubble cannot sustain in equilibrium.

Households.sincs.icelds. cons291(ti1(stand)-(s.)-463(s.)Td[(thsc)-(s.)-402(ass)d)-(s.)-462e.g.ba363(th)27(la

assets. As equation (65) shows, equilibrium only determines the aggregate bubble $B_t + P_t M_t$. But the decomposition is indeterminate because both B_t and P_t satisfy the same equilibrium conditions and follow the same dynamics (see (46) and (64)). The equilibrium real allocation is independent of the decomposition. This result is analogous to that discussed in Section 5 of Tirole (1985).

In this paper, we provide an in…nite-horizon model of a production economy with bubbles, in which …rms meet stochastic investment opportunities and face credit constraints. Capital is not only an input for production, but also serves as collateral. We show that bubbles on this reproducible asset may arise, which relax collateral constraints and improve investment e¢ ciency. The collapse of bubbles leads to a recession, even though there is no exogenous shock to the fundamentals of the economy. Immediately after the collapse, investment falls discontinuously and the stock market crashes in that the stock price falls discontinuously. In the long run, output, investment, consumption, and capital all fall Td[dim1TTd[g4243a)-1(mption)-1(om)2ong

Appendices

Substituting the conjecture (10) into (4) and (6) yields:

$$v_t K_t^j + b_t \quad = \quad \max \; R_t K_t^j dt \qquad I_t^j dt + e^{\;\; rdt} v_{t+dt} \;\; (1 \qquad dt)\, K_t^j + I_t^j \qquad dt \tag{A.1}$$
$$+ e^{\;\; rdt} v_{t+dt}\,(1 \qquad dt)\, K_t^j\,(1 \qquad dt) + B_t;$$

$$L_t^j \qquad e^{\;\; rdt} v_{t+dt} K_t^j + B_t; \tag{A.2}$$

where B_t is de…ned in (12). We combine (5) and (A.2) to obtain:

$$0 \quad I_t^j \qquad R_t K_t^j + \quad e^{\;\; rdt} v_{t+dt} K_t^j + B_t: \tag{A.3}$$

Let Q_t be the Lagrange multiplier associated with (3) for the case with the arrival of the investment opportunity. The …rst-order condition with respect to K_{t+dt}^j delivers equation (13). When $Q_t > 1$; we obtain the optimal investment rule in (11). Plugging (11) and (3) into the Bellman equation (A.1) and matching coe¢ cients of K_t^j and the terms unrelated to K_t^j; we obtain (14) and (15). Q.E.D.

Using the optimal investment rule in (11) and aggregating equation (3), we obtain the aggregate capital accumulation equation (18) and the aggregate investment equation (22). Substituting (15) into (12) yields (16). Substituting (14) into (13) yields (17). The …rst-order condition for the static labor choice problem (1) gives $w_t = (1 \qquad)\,(K_t^j = N_t^j)$. We then obtain (2) and $K_t^j = N_t^j\,(w_t = (1 \qquad))^{1=}$: Thus, the capital-labor ratio is identical for each …rm. Aggregating yields $K_t = N_t\,(w_t = (1 \qquad))^{1=}$: Using this equation to substitute out w_t in (2) yields $R_t = \quad K_t^{\quad 1} N_t^{1} \quad = \quad K_t^{\quad 1}$ since $N_t = 1$ in equilibrium. Aggregate output satis…es

$$Y_t = \int (K_t^j)\;(N_t^j)^{1} \quad dj = \int (K_t^j = N_t^j)\;\; N_t^j dj = (K_t^j = N_t^j) \int N_t^j dj = K_t\, N_t^{1} \quad :$$

This completes the proof. Q.E.D.

By equation (18),

$$\frac{K_{t+dt} \quad K_t}{dt} = \qquad K_t + [R_t + \quad Q_t K_t + B_t] \quad :$$

Taking limit as $dt \to 0$ yields equation (21). Using the approximation $e^{rdt} = 1 + rdt$ in equation (16) yields:

$$B_t(1 + rdt) = B_{t+dt}[1 + (Q_{t+dt} - 1)dt]:$$

Simplifying yields:

$$\frac{B_t - B_{t+dt}}{dt} + rB_t = B_{t+dt}\ (Q_{t+dt} - 1):$$

Taking limits as $dt \to 0$ yields equation (19). Finally, we approximate equation (17) by:

$$Q_t(1 + rdt) = R_{t+dt}dt + (1 - dt)Q_{t+dt} + (R_{t+dt} + Q_{t+dt})(Q_{t+dt} - 1)\ dt:$$

Simplifying yields:

$$\frac{Q_t - Q_{t+dt}}{dt} + rQ_t = R_{t+dt} - Q_{t+dt} + (R_{t+dt} + Q_{t+dt})(Q_{t+dt} - 1)\ :$$

Taking limit as $dt \to 0$ yields equation (20).

We may start with a continuous-time formulation directly. The Bellman equation in continuous time satisfies:

$$rV\ K^j;S = \max_{I^j} RK^j - I^j + V\ K^j + I^j;S - V\ K^j;S$$
$$K^j V_K\ K^j;S + V_S\ K^j;S\ \dot{S};$$

where $S = (B;Q)$ represents the vector of aggregate state variables. We may derive this Bellman equation by taking limits in (4) as $dt \to 0$: Conjecture $V\ K^j;B;Q = QK^j + B$: We can then solve the above Bellman equation. After aggregation, we can derive the system of differential equations in the proposition. Q.E.D.

(i) The social planner solves the following problem:

$$\max_{I_t} \int_0^1 e^{rt}(K_t - I_t)\,dt$$

subject to

$$\dot{K}_t = K_t + I_t;\ K_0 \text{ given}$$

where K_t is the aggregate capital stock and I_t is the investment level for each firm with the arrival of the investment opportunity. From this problem, we can derive the efficient capital stock K_E; which satisfies $(K_E)^{1} = r +$: The efficient output, investment and consumption levels are given by $Y_E = (K_E)$; $I_E = = K_E$; and $C_E = (K_E) - K_E$; respectively.

From the proof of Proposition 1, we can rewrite (A.1) as:

$$v_t K_t^j = \max \; R_t K_t^j dt \quad I_t^j dt + Q_t(1 \quad dt)K_t^j + Q_t \;\; I_t^j dt: \tag{A.4}$$

Suppose assumption (25) holds. We conjecture $Q \;\; = 1$ and $Q_t = 1$. Substituting this conjecture into the above equation and matching coe¢ cients of K_t^j give:

$$v_t = R_t dt + 1 \quad dt:$$

Since $Q_t = e^{\;\; rdt} v_{t+dt} = 1$; we have $e^{rdt} = R_{t+dt}dt + 1 \quad dt$: Approximating this equation yields:

$$1 + rdt = R_{t+dt}dt + 1 \quad dt:$$

Taking limits as $dt \; ! \; 0$ gives $R_t = r + \quad = \quad K_t^{\quad 1}$: Thus, $K_t = K_E$: Given this constant capital stock for all …rms, the optimal investment level satis…es $K_t = \quad I_t$: Thus, $I_t = K_t = \; = \;$: We can easily check that assumption (25) implies that

$$- = I_t = K_t \quad R_t + \quad = r + \quad + \; :$$

Thus, the investment constraint (5) or (A.3) is satis…ed for $Q_t = 1$ and $B_t = 0$: We conclude that the solutions $Q_t = 1$, $K_t = K_E$; and $I_t = K_t = \; = \;$ give the bubbleless equilibrium, which also delivers the e¢ cient allocation.

(ii) Suppose (26) holds. Conjecture $Q_t > 1$ in some neighborhood of the bubbleless steady state. We can then apply Proposition 3 and derive the steady-state equations (23) and (24). From these equation, we obtain the steady-state solution Q and K in (27) and (28), respectively. Assumption (26) implies that $Q \;\; > 1$: By continuity, $Q_t > 1$ in some neighborhood of $(Q \; ; K \;)$: This veri…es our conjecture. Q.E.D.

Solving equations (23), (30), (31) yields equations (32)-(34). By (32), $B \; > 0$ if and only if (35) holds. From (27) and (33), we deduce that $Q_b < Q$: Using condition (35), it is straightforward to check that $K_{GR} > K_E > K_b > K$. From (32), it is also straightforward to verify that the bubble-asset ratio $B = K_b$ decreases with : Q.E.D.

First, we consider the log-linearized system around the bubbly steady state $(B; Q_b; K_b)$: We use $\hat{X}_t$ to denote the percentage deviation from the steady state value for any variable X_t, i.e., $\hat{X}_t = \ln X_t \quad \ln X$: We can show that the log-linearized system is given by:

Since $_1 < 0$ the above equation and equation (A.8) imply that $g > 0$.

From the above analysis, we conclude that the matrix A has one negative eigenvalues and the other two eigenvalues are either positive real numbers or complex numbers with positive real part. As a result, the bubbly steady state is a local saddle point and the stable manifold is one dimensional.

Next, we consider the local dynamics around the bubbleless steady state $(0; Q \;; K \;)$. We linearize B_t around zero and log-linearize Q_t and K_t and obtain linearized system:

$$\begin{bmatrix} dB_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = J \begin{bmatrix} B_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix};$$

where

$$J = \begin{bmatrix} r \quad (Q \quad 1) & 0 & 0 \\ 0 & a & b \\ \overline{K^*} & c & d \end{bmatrix};$$

where

$$\begin{aligned} a &= \frac{R}{Q}[1 + \;(Q \quad 1)] \quad (\frac{R}{Q} + \;)\; Q \;; \\ b &= \frac{R}{Q}[1 + \;(Q \quad 1)](1 \quad\;) > 0; \\ c &= \quad Q \; > 0; \\ d &= \quad R\;[\quad 1] < 0: \end{aligned}$$

Using a similar method for the bubbly steady state, we analyze the three eigenvalues of the matrix J. One eigenvalue, denoted by $_1$; is equal to $r \quad (Q \quad 1) < 0$ and the other two, denoted by $_2$ and $_3$; satisfy

$$_2\;_3 = ad \quad bc: \tag{A.9}$$

Notice that we have

$$\frac{a}{b} = \frac{1}{1} \quad \frac{1}{1 + \;(Q \quad 1)} \quad \frac{Q}{R}\frac{Q}{1 + \;(Q \quad 1)} \;;$$

and

$$\frac{c}{d} = \quad \frac{Q}{R}\frac{1}{1}:$$

So we have

$$\frac{a}{b} \quad \frac{c}{d} > 0 \text{ or } \frac{a}{b} > \frac{c}{d}:$$

Since $b > 0$ and $d < 0$, we deduce that $ad < cb$. It follows from (A.9) that

As we discussed in the main text, we may derive equations (44) and (45). Substituting equation (44) into (42) and using the de…nition $Q_t = e^{\ rdt} v_{t+dt}$; we can derive that:

$$
\begin{aligned}
Q_t \quad = \quad & Q_t dt + e^{\ rdt}(1 \quad dt)[R_{t+dt}dt + Q_{t+dt}(1 \quad dt) \\
& + \ (Q_{t+dt} \quad 1)(R_{t+dt} + Q_{t+dt} \ )dt]: \qquad \text{(A.10)}
\end{aligned}
$$

Using the approximation $e^{\ rdt} = 1 \quad rdt$ and removing all terms that have orders at least dt^2; we approximate the above equation by:

$$
\begin{aligned}
Q_t \quad Q_{t+dt} \quad = \quad & Q_t dt + R_{t+dt}dt \quad Q_{t+dt}dt + \ (Q_{t+dt} \quad 1)(R_{t+dt} + \ Q_{t+dt})dt \\
& (r + \ ) Q_{t+dt}dt: \qquad \text{(A.11)}
\end{aligned}
$$

Dividing by dt on the two sides and taking limits as $dt \ ! \ 0$; we obtain:

$$
Q_t = R_t \quad (r + \quad + \ )Q_t + \ Q_t + \ (Q_t \quad 1)(R_t + \ Q_t); \qquad \text{(A.12)}
$$

which gives equation (47). Similarly, substituting equation (45) into (43) and taking limits, we can derive equation (46). Q.E.D.

Let $Q(\ )$ be the expression on the right-hand side of equation (48). We then use this equation to rewrite equation (49) as:

$$
K \quad ^1(1 + r + \ ) \quad (r + \quad + \ )Q(\ ) + \ g(K) + (r + \ ) \ Q(\ ) = 0:
$$

De…ne the function $F(K; \ )$ as the expression on the left-hand side of the above equation. Notice $Q(\ ) = Q \ = g(K \ )$ by de…nition and $Q(0) = Q_b$ where Q_b is given in (33). The condition (35) ensures the existence of the bubbly steady-state value Q_b and the bubbleless steady-state values Q and K .

De…ne

$$
K_{max} = \max_{0 \quad *} \left[\frac{(r + \quad + \quad (r + \ ) \ )Q(\ ) \quad Q}{(1 + r + \ )} \right]^{\frac{1}{-1}}:
$$

By (34), we can show that

$$
K_b = \frac{(r + \quad r \ )Q(0)}{}
$$

for 2 (0;) : If this true, then it follows from the intermediate value theorem that there exists a solution K_s to $F(K;) = 0$ such that K_s 2 $(K ; K_{max})$:

First, notice that

$$\begin{aligned} F(K ;0) &= K \quad ^{1}(1+r) \quad r(1 \quad)Q_b \quad Q_b \\ &> K_b \ ^{1}(1+r) \quad r(1 \quad)Q_b \quad Q_b \\ &= 0; \end{aligned}$$

and

$$F(K ; \quad) = 0:$$

We can verify that $F(K;)$ is concave in for any ...xed K: Thus, for all $0 < \quad < \quad$;

$$\begin{aligned} F(K ;) &= F \quad K ;(1 \quad —)0 + — \\ &> (1 \quad —)F(K ;0) + —F(K ; \quad) \\ &> 0: \end{aligned}$$

Next, for K 2 $(K ; K_{max})$, we derive the following:

$$\begin{aligned} F(K_{max};) &= K_{max}^{1}(1+r+ \quad) \quad (r+ \quad + \quad)Q(\ ) + \ g(K_{max}) + (r+ \quad) \ Q(\ ) \\ &< K_{max}^{1}(1+r+ \quad) \quad (r+ \quad + \quad)Q(\ ) + \ g(K \ ) + (r+ \quad) \ Q(\ ) \\ &< 0; \end{aligned}$$

where the ...rst inequality follows from the fact that the saddle path for the bubbleless equilibrium is downward sloping as illustrated in Figure 3 so that $g(K_{max}) < g(K \ )$; and the second inequality follows from the de...nition of K_{max} and the fact that $g(K \ ) = Q$:

Finally, note that $Q(\ ) < Q$ for $0 < \quad < \quad$: We use equation (31) and $K_s > K$ to deduce that

$$\begin{aligned} \frac{B_s}{K_s} &= — \quad K_s \ ^{1} \quad Q(\ ) \\ &> — \quad K \ ^{1} \quad Q \\ &= 0: \end{aligned}$$

This completes the proof of the existence of stationary equilibrium with stochastic bubbles $(B_s; Q_s; K_s)$:

more realistic prediction after introducing variable capacity utilization. Assume that each …rm j's production is given by

$$Y_t^j = (u_t^j K_t^j) \quad N_t^j \;\; {}^{1} \quad ; \tag{B.1}$$

where u_t^j represents the capacity utilization rate. Assume that the …rm's deprecation rate between period t to period $t + dt$ is given by

$$ {}_t^j = \ '(u_t^j); \tag{B.2}$$

where ' is an increasing and convex function. Suppose that the capacity utilization decision

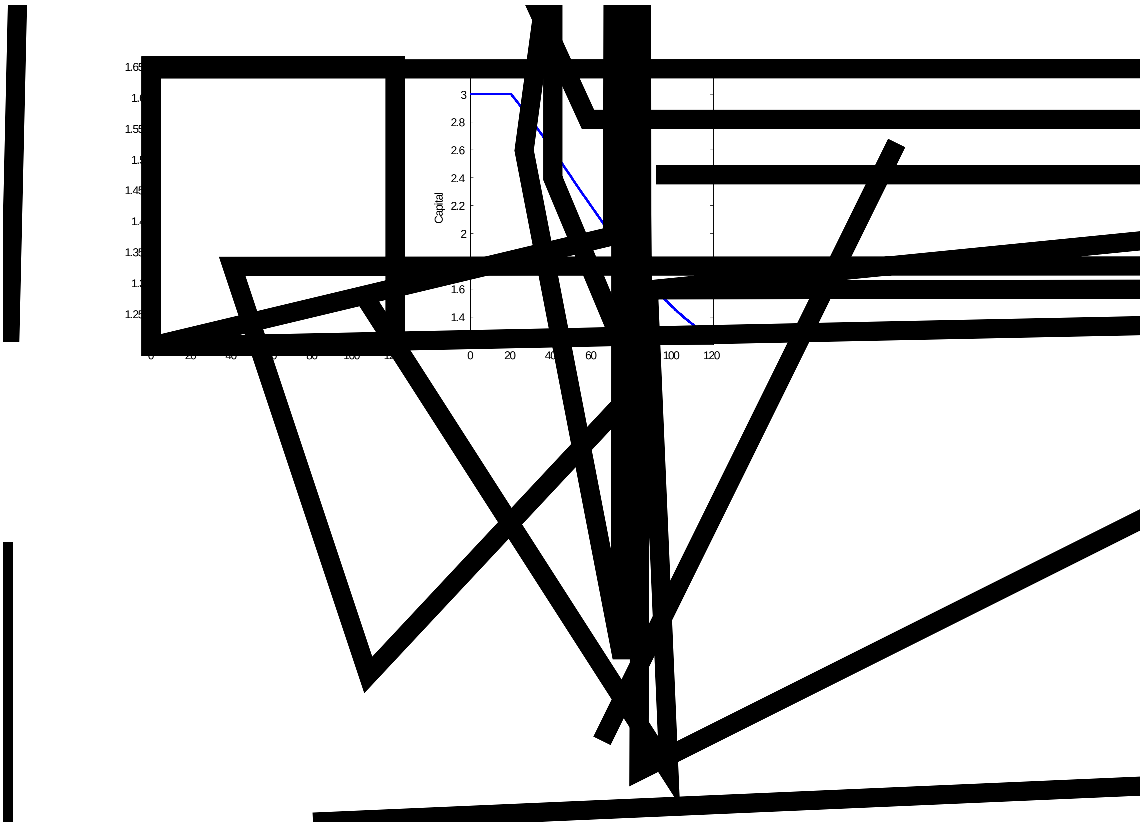

Figure 5: This figure plots the dynamics of the stationary equilibrium with capacity utilization and stochastic bubbles. Set the paramter values as follows: r = 0:02, = 0:4, = 0:01; = 0:05; = 0:2, = 0:4, $_0$ = 0:0075; and $_1$ = 0:0245.

Next, we turn to the equilibrium with stochastic bubbles. When the bubble bursts, the economy will be at the bubbleless equilibrium described in the previous proposition. We write the capital price in a feedback form $Q_t = Q(K_t)$ for some function K_t: As in Section 6, we characterize the equilibrium before the bubble bursts as follows:

$$Q_t > 1:$$

$(B_t; Q_t; K_t; u_t)$

$$Q_t = (r + \quad _t + \quad)Q_t \quad Q_t \quad u_t R_t \quad (Q_t \quad 1)(u_t R_t + \quad Q_t); \tag{B.8}$$

$$B_t = (r + \quad)B_t \quad (Q_t \quad 1)B_t; \tag{B.9}$$

$$K_t = \quad _t K_t + (u_t R_t K_t + \quad Q_t K_t + B_t) \quad ; \tag{B.10}$$

$u_t \quad R_t \quad Q_t = Q(K_t):$

As in Section 6, we can simulate the impact of the collapse of bubbles. In doing so, we specify: $'(u_t) = \ _0 + \ _1 \frac{u_t^{1+}}{1+}$ and take parameter values given in Figure 5. Suppose the economy is in a steady state initially and then the bubble bursts at t = 20. On impact, Q_t immediately jumps and so does the capacity utilization rate. Since one unit of installed capital becomes more valuable, the capacity utilization rate decreases to slow down depreciation. As a result, both output and consumption drop immediately. Figure 5 reveals that the economy enters into a prolonged recession after the burst of bubbles.

In the base model studied in Section 2, we have considered intratemporal debt and assumed that …rms cannot borrow and save intertemporally. In this appendix, we shall relax this assumption. We assume that there is no intratemporal debt and …rms can borrow and save intertemporally by trading risk-free one-period private bonds. Households can also trade these bonds, but face short sales constraints. Intuitively, each period …rms without investment opportunities have extra funds and may lend to …rms with investment opportunities. They may also save their funds and use these funds to …nance investments when investment opportunities arrive in the future. We will show that bubbles on …rm assets can still exist even though the steady-state interest rate on the private bonds is positive. In addition, our key insights still carry over to this setup.

Consider a discrete-time setup outlined in Section 2 and time is denoted by $t = 0; 1; 2::::$ Let $= e^{r}$: Let R_{ft} denote the gross interest rate of the bonds. Let L_t^h denote households' bond holdings. The short sales constraint is given by $L_t^h \quad 0$ for all t:

Let $V_{1t}\left(K_t^j; L_t^j\right)$ $(V_{0t}(K_t^j; L_t^j))$ denote the stock market value of a typical ...rm j with (without) investment opportunities when its capital stock and debt at time t are K_t^j and L_t^j; respectively. We suppress the aggregate state variables in the argument. When $L_t^j < 0$; L_t^j means savings. Then $V_{1t}\left(K_t^j; L_t^j\right)$ satis...es the Bellman equation:

$$V_{1t}\left(K_t^j; L_t^j\right) = \max_{I_t^j; L_{t+1}^j} R_t K_t^j \quad I_t^j + \frac{L_{t+1}^j}{R_{ft}} \quad L_t^j \\ + \left[V_{1t+1}\left(K_{t+1}^j; L_{t+1}^j\right) + (1 \quad) V_{0t+1}\left(K_{t+1}^j; L_{t+1}^j\right) \right]; \tag{C.1}$$

subject to

$$K_{t+1}^j = (1 \quad) K_t^j + I_t^j; \tag{C.2}$$

$$D_t^j + I_t^j = R_t K_t^j + \frac{L_{t+1}^j}{R_{ft}} \quad L_t^j; \tag{C.3}$$

$$D_t^j \quad (1 \quad) L_t^j; \tag{C.4}$$

and a borrowing constraint (C.6) described below. Equation (C.3) is a ‡ow of funds constraint, where D_t^j means dividend payments if $D_t^j \quad 0$ and new equity if $D_t^j < 0$: We assume that external equity ...nance is costly so that the value of new equity is limited by a fraction $(1 \quad)$ of debt as shown in (C.4). This constraint ensures that new equity is not su¢ cient to ...nance investment and ...rms still need debt ...nancing. If $= 1$; then ...rms cannot raise any new equity.

A ...rm without investment opportunities solves the following dynamic programming problem:

$$V_{0t}\left(K_t^j; L_t^j\right) = \max_{L_{t+1}^j} R_t K_t^j + \frac{L_{t+1}^j}{R_{ft}} \quad L_t^j \\ + \left[V_{1t+1}\left((1 \quad) K_t^j; L_{t+1}^j\right) + (1 \quad) V_{0t+1}\left((1 \quad) K_t^j; L_{t+1}^j\right) \right]; \tag{C.5}$$

subject to a borrowing constraint (C.6) described below and a constraint on new equity. We will show below that both constraints will not bind in equilibrium.

We now introduce the credit constraint:

$$E_t V_{t+1}\left(K_{t+1}^j; L_{t+1}^j\right) \quad E_t V_{t+1}\left(K_{t+1}^j; 0\right) \quad E_t V_{t+1}(\ K_t^j; 0); \tag{C.6}$$

where we de…ne

$$E_t V_{t+1}$$

and $v_{0t} = R_t + (1 \quad)Q_t$, $b_{0t} = B_t$ and $v^L_{0t} = 1$.

Next, we consider problem (C.1) for a …rm with investment opportunities. Substituting conjecture (C.7) into this problem yields:

$$v_{1t}K_t^j \quad v^L_{1t}L_t^j + b_{1t} = \max_{I_t^j;L_{t+1}^j} R_tK_t^j \quad I_t^j + \frac{L_{t+1}^j}{R_{ft}} \quad L_t^j \qquad \text{(C.11)}$$
$$+Q_t[(1 \quad)K_t^j + I_t^j] \quad Q_t^L L_{t+1}^j + B_t:$$

If $Q_t > 1$ then both the borrowing constraint (C.8) and the constraint (C.4) must bind. Using these constraints as well as (C.10) and (C.3), we obtain:

$$I_t^j = R_tK_t^j + Q_t \;\; K_t^j + B_t \qquad L_t^j: \qquad \text{(C.12)}$$

Substituting this equation back into the Bellman equation (C.11) and matching coe¢ cients, we obtain:

$$v^L_{1t} = 1 + \quad (Q_t \quad 1); \qquad \text{(C.13)}$$

$$v_{1t} = \;\; R_t + (1 \quad)Q_t + (Q_t \quad 1)(\;\; Q_t + R_t); \qquad \text{(C.14)}$$

$$b_{1t} = B_t + (Q_t \quad 1)B_t: \qquad \text{(C.15)}$$

By de…nition of B_t and Q_t introduced in this appendix, we can show that:

$$B_t = \quad B_{t+1}(1 + \quad (Q_{t+1} \quad 1); \qquad \text{(C.16)}$$

$$Q_t = \quad [R_{t+1} + (1 \quad)Q_{t+1} + \quad (Q_{t+1} \quad 1)(\;\; Q_{t+1} + R_{t+1})]: \qquad \text{(C.17)}$$

The aggregate capital stock satis…es:

$$K_{t+1} = (1 \quad)K_t + \quad (R_tK_t + Q_t \;\; K_t + B_t) \qquad \int L_t^j dj$$
$$= (1 \quad)K_t + \quad (R_tK_t + Q_t \;\; K_t + B_t); \qquad \text{(C.18)}$$

where the second line follows from the bond market-clearing condition $\int L_t^j dj = 0$, when households do not hold any bonds.

By de…nition of Q_t^L; $v^L_{0t} = 1$, and equations (C.10) and (C.14), we can show that:

$$\frac{1}{R_{ft}} = Q_t^L = \quad v^L_{1t} + (1 \quad)v^L_{0t} = \quad [1 + \quad (Q_{t+1} \quad 1)]: \qquad \text{(C.19)}$$

If $Q_{t+1} > 1$, then

$$R_{ft} = \frac{1}{1 + \quad (Q_{t+1} \quad 1)} < 1: \qquad \text{(C.20)}$$

Thus, households prefer to sell bonds until their borrowing constraints bind so that $L_t^h = 0$ for all t: Because …rms with investment opportunities choose to borrow and invest, for the bond market to clear, …rms without investment opportunities must save and lend. By (C.16) and (C.20), we …nd that $R_{ft} > B_{t+1}=B_t$ for $\;2\;(0;1)$ and $B_t > 0$ so that the interest rate of bonds is larger than the growth rate of bubbles. Only when $\;= 1;\; R_{ft} = B_{t+1}=B_t$:

Notice that equations (C.16)-(C.18) are exactly the same as equations (16)-(18) for dt = 1 and r = log( ). Thus, our analyses of bubbleless and bubbly equilibria in Sections 4-5 can be carried over to the present setup, by suitably translating the continuous-time results into the discrete time results.

In the steady state, $B_t = B$ is constant over time, where B may be zero or positive. But the steady-state net rate of interest rate on bonds $R_f \;\; 1$ is positive. The steady state aggregate dividends D satisfy:

$$D = RK \quad I = RK \quad (RK + Q\;K + B):$$

By (C.18), the steady state capital stock K satis…es:

$$K = \;(RK + Q\;K + B):$$

Combining the above two equations yields $D = (R \quad)\,K > 0$; where the inequality follows from (28) or (34). Because …rms with investment opportunities borrow and raise new equity to …nance investment, …rms without investment opportunities must save and payout dividends for the aggregate dividends D to be positive. This result is true in the neighborhood of the steady state. Thus, the constraint on new equity …nance for …rms without investment opportunities will never bind, con…rming our previous claim.

In the main text, we have assumed that investment opportunities arrive stochastically according to a Poisson process. In this appendix, we modify our model by considering the case where …rms are subject to idiosyncratic investment-speci…c shocks with a continuous distribution in a discrete time setup. We will show that our key insights carry over to this case. In addition, we will show that stock-price bubbles can arise even though capital can be fully pledgeable, e.g, = 1.

We replace equation (3) with

$$K_{t+1}^j = (1 \quad)\,K_t^j + "_t^j I_t^j; \qquad \text{(D.1)}$$

where ε_t^j represents idiosyncratic investment-specific shocks. Assume that ε_t^j is independently and identically distributed over time and across firms. It is drawn from a continuous distribution function with the support $[\varepsilon_{min}; \varepsilon_{max}]$; where $\varepsilon_{min} \quad 0$:

Let $V_t \left(K_t^j; \varepsilon_t^j\right)$ denote the cum-dividends stock value after the idiosyncratic shock ε_t^j is realized. It satisfies the following Bellman equation:

$$V_t(K_t^j; \varepsilon_t^j) = \max_{I_t^j} \; R_t K_t^j \quad I_t^j + \quad E_t V_{t+1}(K_{t+1}^j; \varepsilon_{t+1}^j); \tag{D.2}$$

subject to the investment constraint (5) and the credit constraint:

$$L_t^j \qquad E_t V_{t+1}(\; K_t^j; \varepsilon_{t+1}^j): \tag{D.3}$$

Here E_t is the conditional expectation operator with respect to ε_{t+1}^j: We conjecture that firm value takes the following form:

$$V_t \left(K_t^j; \varepsilon_t^j\right) = v_t \left(\varepsilon_t^j\right) K_t^j + b_t \left(\varepsilon_t^j\right);$$

where $v_t \left(\varepsilon_t^j\right)$ and $b_t \left(\varepsilon_t^j\right)$ are functions to be determined.

Following a similar proof for Proposition 1, we can show that optimal investment is given by:

$$I_t^j = \begin{cases} 0 & \varepsilon_t^j < 1/Q_t \\ R_t K_t^j + \quad Q_t K_t^j + B_t & \varepsilon_t^j \quad 1/Q_t \end{cases}; \tag{D.4}$$

where

$$B_t = \int b_{t+1}(\varepsilon)\, d \quad (\varepsilon); \quad Q_t = \int v_{t+1}(\varepsilon)\, d \quad (\varepsilon);$$

$$v_t(\varepsilon_t^j) = \begin{cases} R_t dt + (1 \quad dt)\, Q_t & \varepsilon_t^j < 1/Q_t \\ (1 \qquad)\, Q_t + (R_t + \quad Q_t)\, Q_t \varepsilon_t^j & \varepsilon_t^j \quad 1/Q_t \end{cases};$$

$$b_t(\varepsilon_t^j) = \begin{cases} B_t & \varepsilon_t^j \quad 1/Q_t \\ B_t Q_t \varepsilon_t^j & \varepsilon_t^j > 1/Q_t \end{cases}:$$

Clearly, when the investment-specific shock ε_t^j is small enough, the firm chooses to reduce investment as much as possible so that investment reaches the lower bound. When the investment-specific shock ε_t^j is large enough, the firm chooses to increase investment until reaching the upper bound.

Following a similar proof for Proposition 2, we can then derive the equilibrium system. We then take the continuous time limit to obtain the following result.

$$(B_t; Q_t; K_t)$$

$$B_t = \quad B_{t+1} \quad \int \max(\frac{"}{"_{t+1}}; 1) \quad (")d" \tag{D.5}$$

$$Q_t = \quad " \quad K_{t+1}^{\quad 1} + (1 \quad)Q_{t+1} + \int_{"^*_{t+1}} \quad K_{t+1}^{\quad 1} + \quad Q_{t+1} \quad (Q_{t+1}" \quad 1) \quad (") \, d" \; \# \tag{D.6}$$

$$K_{t+1} = (1 \quad)K_t + (\quad K_t \quad + K_t \quad Q_t + B_t) \int_{"^*_t}^{1} " \quad (")d" \tag{D.7}$$

$$"_t = \frac{1}{Q_t} \tag{D.8}$$

$$\lim_{T \,!\, 1} \quad {}^T Q_T K_{T+1} = 0 \quad \lim_{T \,!\, 1} \quad {}^T B_{T+1} = 0:$$

Clearly, there is one equilibrium without bubbles in which $B_t = 0$ for all t: To study the existence of a bubbly equilibrium, we consider the Pareto distribution $(") = 1 \quad "$ where > 1 and $" \quad 1$: We shall focus on the steady state. In the bubbly steady state, B_t; Q_t and K_t are equal to some constants $B; Q_b$ and K_b over time, respectively. We can then use the above proposition to show that:

$$Q_b = \quad (\quad 1) \quad {}^1 \quad 1 \quad {}^{\frac{1}{}};$$

$$R_b = Q_b[1 \quad (1 \quad) + (\quad 1) \quad] > 0;$$

$$\frac{B}{K_b} = \frac{}{\int_{"^*_b}^{1} " \quad (")d"} \quad R_b \quad Q_b$$

$$= \quad Q_b \quad \frac{}{{}^1 \quad 1} \quad 1 + \quad (1 \quad) \quad ! :$$

For $B > 0$; we need

$$< \frac{}{(1 \quad)} + 1 \quad \frac{1}{} \quad :$$

This is the condition for the existence of bubbles for the model in this appendix. We can see that if

$$1 < \quad <$$

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